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STRONG TAYLOR APPROXIMATION OF STOCHASTIC DIFFERENTIAL EQUATIONS AND APPLICATION TO THE LÉVY LIBOR MODEL

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Abstract. In this article we consider the strong approximation of stochastic differential equations driven by Lévy processes or general semimartingales. The main ingredients of our method is the perturbation of the SDE and the Taylor expansion of the resulting parameterized curve. We apply this method to develop strong approximation schemes for LIBOR market models. In particular, we derive fast and precise algorithms for the valuation of derivatives in LIBOR models which are more tractable than the simulation of the full SDE. A numerical example for the Lévy LIBOR model illustrates our method.

1. Introduction

The main aim of this paper is to develop a general method for the strong approximation of stochastic differential equations (SDEs) and to apply it to the valuation of options in LIBOR models. The method is based on the perturbation of the initial SDE by a real parameter ϵ and then on the Taylor expansion of the resulting parameterized curve around $\epsilon = 0$. Thus, we follow the line of thought of Siopacha and Teichmann (2007) and extend their results from continuous to general semimartingales. The motivation for this work comes from LIBOR market models; in particular, we consider the Lévy LIBOR model as the basic paradigm for the development of this method.

The LIBOR market model has become a standard model for the pricing of interest rate derivatives in recent years. The main advantage of the LIBOR model in comparison to other approaches is that the evolution of discretely compounded, market-observable forward rates is modeled directly and not deduced from the evolution of unobservable factors. Moreover, the log-normal LIBOR model is consistent with the market practice of pricing caps according to Black's formula (cf. Black 1976). However, despite its apparent popularity, the LIBOR market model has certain well-known pitfalls.

On the one hand, the log-normal LIBOR model is driven by a Brownian motion, hence it cannot be calibrated adequately to the observed market data. An interest rate model is typically calibrated to the implied volatility

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surface from the cap market and the correlation structure of at-the-money swaptions. Several extensions of the LIBOR model have been proposed in the literature using jump-diffusions, Lévy processes or general semimartingales as the driving motion (cf. e.g. Glasserman and Kou 2003, Eberlein and Özkan 2005, Jamshidian 1999), or incorporating stochastic volatility effects (cf. e.g. Andersen and Brotherton-Ratcliffe 2005).

On the other hand, the dynamics of LIBOR rates are *not* tractable under every forward measure due to the random terms that enter the dynamics of LIBOR rates during the construction of the model. In particular, when the driving process has continuous paths the dynamics of LIBOR rates are tractable under their corresponding forward measure, but they are not tractable under any other forward measure. When the driving process is a general semimartingale, then the dynamics of LIBOR rates are not even tractable under their very own forward measure. Consequently:

(1) if the driving process is a *continuous* semimartingale caplets can be priced in closed form, but *not* swaptions or other multi-LIBOR derivatives;
(2) if H is a *general* semimartingale, then even caplets *cannot* be priced in closed form.

The standard remedy to this problem is the so-called "frozen drift" approximation, where one replaces the random terms in the dynamics of LIBOR rates by their deterministic initial values; it was first proposed by Brace et al. (1997) for the pricing of swaptions and has been used by several authors ever since. Brace et al. (2001), Dun et al. (2001) and Schlögl (2002) argue that freezing the drift is justified, since the deviation from the original equation is small in several measures.

The frozen drift approximation is by far the simplest approximation possible, however it is well-known that it does not yield acceptable results, especially for exotic derivatives and longer horizons. Therefore, several other approximations have been developed in the literature; in one line of research Daniluk and Gątarek (2005) and Kurbanmuradov et al. (2002) are looking for the best lognormal approximation of the forward LIBOR dynamics; cf. also Schoenmakers (2005). Other authors have been using linear interpolations and predictor-corrector Monte Carlo methods to get a more accurate approximation of the drift term (cf. e.g. Pelsser et al. 2005, Hunter et al. 2001 and Glasserman and Zhao 2000). We refer the reader to Joshi and Stacey (2008) for a detailed overview of that literature, and for some new approximation schemes and numerical experiments.

Although most of this literature focuses on the lognormal LIBOR market model, Glasserman and Merener (2003b, 2003a) have developed approximation schemes for the pricing of caps and swaptions in jump-diffusion LIBOR market models.

In this article we develop a general method for the approximation of the random terms that enter into the drift of LIBOR models. In particular, by perturbing the SDE for the LIBOR rates and applying Taylor's theorem we develop a generic approximation scheme; we concentrate here on the first order Taylor expansion, although higher order expansions can be considered in the same framework. At the same time, the frozen drift approximation

can be embedded in this method as the zero-order Taylor expansion, thus offering a theoretical justification for this approximation. The method we develop yields more accurate results than the frozen drift approximation, while being computationally simpler than the simulation of the full SDE for the LIBOR rates.

The method we develop is universal and can be applied to any LIBOR model driven by a general semimartingale. However, we focus on the Lévy LIBOR model as a characteristic example of a LIBOR model driven by a general semimartingale.

The article is structured as follows: in section 2 we review time-inhomogeneous Lévy process, and in section 3 we revisit the Lévy LIBOR model. In section 4 we derive the dynamics of LIBOR rates under the terminal martingale measure and express them as a Lévy-driven SDE. In section 5 we develop the strong Taylor approximation method and apply it to the Lévy LIBOR model. Finally, section 6 contains a numerical illustration.

2. Time-inhomogeneous Lévy processes

Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a complete stochastic basis, where $\mathcal{F} = \mathcal{F}_{T_*}$ and the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T_*]}$ satisfies the usual conditions; we assume that $T_* \in \mathbb{R}_{\geqslant 0}$ is a finite time horizon. The driving process $H = (H_t)_{0\leq t\leq T_*}$ is a *time-inhomogeneous Lévy process*, or a *process* with *independent increments* and *absolutely continuous* characteristics. That is, H is an adapted, càdlàg, real-valued stochastic process with independent increments, starting from zero, where the law of H_t, $t \in [0, T_*]$, is described by the characteristic function

$$\mathbb{E}\left[\mathrm{e}^{iuH_t}\right] = \exp\left(\int_0^t \Big[ib_s u - \frac{c_s}{2}u^2 + \int_{\mathbb{R}} (\mathrm{e}^{iux} - 1 - iux)F_s(\mathrm{d}x)\Big]\mathrm{d}s\right); \quad (2.1)$$

here $b_t \in \mathbb{R}$, $c_t \in \mathbb{R}_{\geqslant 0}$ and F_t is a Lévy measure, i.e. satisfies $F_t(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)F_t(\mathrm{d}x) < \infty$, for all $t \in [0, T_*]$. In addition, the process H satisfies Assumptions ($\mathbb{AC}$) and ($\mathbb{EM}$) given below.

Assumption ($\mathbb{AC}$). The triplets (b_t, c_t, F_t) satisfy

$$\int_0^{T_*} \left(|b_t| + c_t + \int_{\mathbb{R}} (1 \wedge |x|^2)F_t(\mathrm{d}x)\right)\mathrm{d}t < \infty. \quad (2.2)$$

Assumption ($\mathbb{EM}$). There exist constants $M, \varepsilon > 0$ such that for every $u \in [-(1+\varepsilon)M, (1+\varepsilon)M] =: \mathbb{M}$

$$\int_0^{T_*} \int_{\{|x|>1\}} \mathrm{e}^{ux} F_t(\mathrm{d}x)\mathrm{d}t < \infty. \quad (2.3)$$

Moreover, without loss of generality, we assume that $\int_{\{|x|>1\}} \mathrm{e}^{ux} F_t(\mathrm{d}x) < \infty$ for all $t \in [0, T_*]$ and $u \in \mathbb{M}$.

These assumptions render the process $H = (H_t)_{0\leq t\leq T_*}$ a *special* semimartingale, therefore it has the canonical decomposition (cf. Jacod and

Shiryaev 2003, II.2.38, and Eberlein et al. 2005)

$$H = \int_0^{\cdot} b_s \mathrm{d}s + \int_0^{\cdot} \sqrt{c_s} \mathrm{d}W_s + \int_0^{\cdot}\int_{\mathbb{R}} x(\mu^H - \nu)(\mathrm{d}s, \mathrm{d}x), \tag{2.4}$$

where μ^H is the random measure of jumps of the process H, ν is the $\mathbb{P}$-compensator of μ^H, and $W = (W_t)_{0\leq t\leq T_*}$ is a $\mathbb{P}$-standard Brownian motion. The *triplet of predictable characteristics* of H with respect to the measure $\mathbb{P}$, $\mathbb{T}(H|\mathbb{P}) = (B, C, \nu)$, is

$$B = \int_0^{\cdot} b_s \mathrm{d}s, \qquad C = \int_0^{\cdot} c_s \mathrm{d}s, \qquad \nu([0,\cdot] \times A) = \int_0^{\cdot}\int_A F_s(\mathrm{d}x)\mathrm{d}s, \tag{2.5}$$

where $A \in \mathcal{B}(\mathbb{R})$; the triplet (b, c, F) represents the *local characteristics* of H. In addition, the triplet of predictable characteristics (B, C, ν) determines the distribution of H, as the Lévy–Khintchine formula (2.1) obviously dictates.

We denote by κ_s the *cumulant generating function* associated to the infinitely divisible distribution with Lévy triplet (b_s, c_s, F_s), i.e. for $z \in \mathbb{M}$, $s \in [0, T]$

$$\kappa_s(z) := b_s z + \frac{c_s}{2} z^2 + \int_{\mathbb{R}} (\mathrm{e}^{zx} - 1 - zx) F_s(\mathrm{d}x). \tag{2.6}$$

Using Assumption ($\mathbb{EM}$) we can extend κ_s to the complex domain $\mathbb{C}$, for $z \in \mathbb{C}$ with $\Re z \in \mathbb{M}$, and the characteristic function of H_t can be written as

$$\mathbb{E}\left[\mathrm{e}^{iuH_t}\right] = \exp\left(\int_0^t \kappa_s(iu)\mathrm{d}s\right). \tag{2.7}$$

If H is a Lévy process, i.e. time-homogeneous, then (b_s, c_s, F_s) and thus also κ_s do not depend on s. In that case, κ equals the cumulant (log-moment) generating function of H_1.

3. The Lévy LIBOR model

The Lévy LIBOR model has been developed by Eberlein and Özkan (2005), following the seminal articles of Sandmann et al. (1995) and Brace et al. (1997) on LIBOR market models driven by Brownian motion; see also Glasserman and Kou (2003) and Jamshidian (1999) for LIBOR models driven by jump processes. The Lévy LIBOR model is a *market model* where the forward LIBOR rate is modeled directly, and is driven by a time-inhomogeneous Lévy process.

Let $0 = T_0 < T_1 < \cdots < T_N < T_{N+1} = T_*$ denote a discrete tenor structure where $\delta_i = T_{i+1} - T_i$, $i \in \{0, 1, \ldots, N\}$. Consider a complete stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P}_{T_*})$ and a time-inhomogeneous Lévy process $H = (H_t)_{0\leq t\leq T_*}$ satisfying Assumptions ($\mathbb{AC}$) and ($\mathbb{EM}$). The process H has predictable characteristics $(0, C, \nu^{T_*})$ or local characteristics $(0, c, F^{T_*})$, and its

canonical decomposition is

$$H = \int_0^{\cdot} \sqrt{c_s}\mathrm{d}W_s^{T_*} + \int_0^{\cdot}\int_{\mathbb{R}} x(\mu^H - \nu^{T_*})(\mathrm{d}s, \mathrm{d}x), \tag{3.1}$$

where W^{T_*} is a $\mathbb{P}_{T_*}$-standard Brownian motion, μ^H is the random measure associated with the jumps of H and ν^{T_*} is the $\mathbb{P}_{T_*}$-compensator of μ^H. We further assume that the following conditions are in force.

(LR1): For any maturity T_i there exists a bounded, continuous, deterministic function $\lambda(\cdot, T_i) : [0, T_i] \to \mathbb{R}$, which represents the volatility of the forward LIBOR rate process $L(\cdot, T_i)$. Moreover,

$$\sum_{i=1}^{N} \left|\lambda(s, T_i)\right| \leq M,$$

for all $s \in [0, T_*]$, where M is the constant from Assumption $(\mathbb{EM})$, and $\lambda(s, T_i) = 0$ for all $s > T_i$.

(LR2): The initial term structure $B(0, T_i)$, $1 \leq i \leq N + 1$, is strictly positive and strictly decreasing. Consequently, the initial term structure of forward LIBOR rates is given, for $1 \leq i \leq N$, by

$$L(0, T_i) = \frac{1}{\delta_i}\left(\frac{B(0, T_i)}{B(0, T_i + \delta_i)} - 1\right) > 0. \tag{3.2}$$

The construction starts by postulating that the dynamics of the forward LIBOR rate with the longest maturity $L(\cdot, T_N)$ is driven by the time-inhomogeneous Lévy process H and evolve as a martingale under the terminal forward measure $\mathbb{P}_{T_*}$. Then, the dynamics of the LIBOR rates for the preceding maturities are constructed by backward induction; they are driven by the same process H and evolve as martingales under their associated forward measures.

Let us denote by $\mathbb{P}_{T_{i+1}}$ the forward measure associated to the settlement date T_{i+1}, $i \in \{0, \ldots, N\}$. The dynamics of the forward LIBOR rate $L(\cdot, T_i)$, for an arbitrary T_i, is given by

$$L(t, T_i) = L(0, T_i) \exp\left(\int_0^t b^L(s, T_i)\mathrm{d}s + \int_0^t \lambda(s, T_i)\mathrm{d}H_s^{T_{i+1}}\right), \tag{3.3}$$

where $H^{T_{i+1}}$ is a special *semimartingale* with canonical decomposition

$$H_t^{T_{i+1}} = \int_0^t \sqrt{c_s}\mathrm{d}W_s^{T_{i+1}} + \int_0^t\int_{\mathbb{R}} x(\mu^H - \nu^{T_{i+1}})(\mathrm{d}s, \mathrm{d}x). \tag{3.4}$$

Here $W^{T_{i+1}}$ is a $\mathbb{P}_{T_{i+1}}$-standard Brownian motion and $\nu^{T_{i+1}}$ is the $\mathbb{P}_{T_{i+1}}$-compensator of μ^H. The dynamics of an arbitrary LIBOR rate again evolves

as a martingale under its corresponding forward measure; therefore, we specify the drift term of the forward LIBOR process $L(\cdot, T_i)$ as

$$b^L(s,T_i) = -\frac{1}{2}\lambda^2(s,T_i)c_s - \int_{\mathbb{R}} \left(\mathrm{e}^{\lambda(s,T_i)x} - 1 - \lambda(s,T_i)x\right) F_s^{T_{i+1}}(\mathrm{d}x). \tag{3.5}$$

The forward measure $\mathrm{I\!P}_{T_{i+1}}$, which is defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T_{i+1}})$, is related to the terminal forward measure $\mathrm{I\!P}_{T_*}$ via

$$\frac{\mathrm{d}\mathrm{I\!P}_{T_{i+1}}}{\mathrm{d}\mathrm{I\!P}_{T_*}} = \prod_{l=i+1}^{N} \frac{1+\delta_l L(T_{i+1},T_l)}{1+\delta_l L(0,T_l)} = \frac{B(0,T_*)}{B(0,T_{i+1})} \prod_{l=i+1}^{N} \left(1+\delta_l L(T_{i+1},T_l)\right). \tag{3.6}$$

The $\mathrm{I\!P}_{T_{i+1}}$-Brownian motion $W^{T_{i+1}}$ is related to the $\mathrm{I\!P}_{T_*}$-Brownian motion via

$$\begin{aligned} W_t^{T_{i+1}} &= W_t^{T_{i+2}} - \int_0^t \alpha(s,T_{i+1})\sqrt{c_s}\mathrm{d}s = \dots \\ &= W_t^{T_*} - \int_0^t \left(\sum_{l=i+1}^{N} \alpha(s,T_l)\right)\sqrt{c_s}\mathrm{d}s, \end{aligned} \tag{3.7}$$

where

$$\alpha(t,T_l) = \frac{\delta_l L(t-,T_l)}{1+\delta_l L(t-,T_l)}\lambda(t,T_l). \tag{3.8}$$

The $\mathrm{I\!P}_{T_{i+1}}$-compensator of μ^H, $\nu^{T_{i+1}}$, is related to the $\mathrm{I\!P}_{T_*}$-compensator of μ^H via

$$\begin{aligned} \nu^{T_{i+1}}(\mathrm{d}s,\mathrm{d}x) &= \beta(s,x,T_{i+1})\nu^{T_{i+2}}(\mathrm{d}s,\mathrm{d}x) = \dots \\ &= \left(\prod_{l=i+1}^{N} \beta(s,x,T_l)\right)\nu^{T_*}(\mathrm{d}s,\mathrm{d}x), \end{aligned} \tag{3.9}$$

where

$$\beta(t,x,T_l,) = \frac{\delta_l L(t-,T_l)}{1+\delta_l L(t-,T_l)}\left(\mathrm{e}^{\lambda(t,T_l)x} - 1\right) + 1. \tag{3.10}$$

Remark 3.1. Notice that the process $H^{T_{i+1}}$, driving the forward LIBOR rate $L(\cdot, T_i)$, and $H = H^{T_*}$ have the same *martingale* parts and differ only in the *finite variation* part (drift). An application of Girsanov's theorem for semimartingales yields that the $\mathrm{I\!P}_{T_{i+1}}$-finite variation part of H is

$$\int_0^{\cdot} c_s \sum_{l=i+1}^{N} \alpha(s,T_l)\mathrm{d}s + \int_0^{\cdot}\int_{\mathbb{R}} x\left(\prod_{l=i+1}^{N} \beta(s,x,T_l) - 1\right)\nu^{T_*}(\mathrm{d}s,\mathrm{d}x).$$

Remark 3.2. The process $H = H^{T_*}$ driving the most distant LIBOR rate $L(\cdot, T_N)$ is – by assumption – a time-inhomogeneous Lévy process. However, this is *not* the case for any of the processes $H^{T_{i+1}}$ driving the remaining

LIBOR rates, because the *random* terms $\frac{\delta_l L(t-,T_l)}{1+\delta_l L(t-,T_l)}$ enter into the compensators $\nu^{T_{i+1}}$ during the construction; see equations (3.9) and (3.10)

4. Dynamics under the Terminal Measure

In this section we derive the stochastic differential equation (SDE) that the dynamics of an arbitrary LIBOR rate satisfy under the terminal measure $\mathbb{P}_{T_*}$; this will be the starting point for the perturbation of the dynamics and the Taylor expansion in the next section. Of course, we could consider the SDE as the defining point for the model, as is often the case in stochastic volatility LIBOR models, cf. e.g. Andersen and Brotherton-Ratcliffe (2005).

Starting with the dynamics of the LIBOR rate $L(\cdot,T_i)$ under the forward martingale measure $\mathbb{P}_{T_{i+1}}$, and using the connection between the forward and terminal martingale measures (cf. eqs. (3.7)–(3.10) and Remark 3.1), we have that the dynamics of the LIBOR rate $L(\cdot,T_i)$ under the terminal measure are given by

$$L(t,T_i) = L(0,T_i)\exp\left(\int_0^t \bar{b}(s,T_i)\mathrm{d}s + \int_0^t \lambda(s,T_i)\mathrm{d}H_s\right), \tag{4.1}$$

where $H = (H_t)_{0\le t\le T_*}$ is the $\mathbb{P}_{T_*}$-time-inhomogeneous Lévy process driving the LIBOR rates, cf. (3.1); the drift term $\bar{b}(\cdot,T_i)$ has the form

$$\begin{aligned}\bar{b}(s,T_i) = &-\frac{1}{2}\lambda^2(s,T_i)c_s - c_s\lambda(s,T_i)\sum_{l=i+1}^{N}\frac{\delta_l L(s-,T_l)}{1+\delta_l L(s-,T_l)}\lambda(s,T_l)\\ &-\int_{\mathbb{R}}\left(\left(\mathrm{e}^{\lambda(s,T_i)x}-1\right)\prod_{l=i+1}^{N}\beta(s,x,T_l) - \lambda(s,T_i)x\right)F_s^{T_*}(\mathrm{d}x),\end{aligned} \tag{4.2}$$

where $\beta(s,x,T_l)$ is given by (3.10). Note that the drift term of (4.1) is random, therefore we are dealing with a general semimartingale, and not with a Lévy process. Of course, $L(\cdot,T_i)$ is not a $\mathbb{P}_{T_*}$-martingale, unless $j=1$ (where we use the conventions that $\sum_{l=1}^{0} = 0$ and $\prod_{l=1}^{0} = 1$).

Let us denote by $\overline{Z}$ the semimartingale that drives the LIBOR rate $L(\cdot,T_i)$ under the terminal measure, i.e.

$$\begin{aligned}\overline{Z} := &\int_0^{\cdot}\bar{b}(s,T_i)\mathrm{d}s + \int_0^{\cdot}\lambda(s,T_i)\sqrt{c_s}\mathrm{d}W_s^{T_*}\\ &+\int_0^{\cdot}\int_{\mathbb{R}}\lambda(s,T_i)x(\mu^H-\nu^{T_*})(\mathrm{d}s,\mathrm{d}x).\end{aligned} \tag{4.3}$$

The triplet of predictable characteristics of $\overline{Z} = (\overline{Z}_t)_{0\le t\le T_*}$ under $\mathbb{P}_{T_*}$, $\mathbb{T}(\overline{Z}|\mathbb{P}_{T_*}) = (\overline{B}, \overline{C}, \overline{\nu})$, is described by

$$\begin{aligned}
\overline{B} &= \int_0^{\cdot} \bar{b}(s,T_i)\mathrm{d}s \\
\overline{C} &= \int_0^{\cdot} \lambda^2(s,T_i)c_s\mathrm{d}s \\
1_A(x) * \overline{\nu} &= 1_A\big(\lambda(s,T_i)x\big) * \nu^{T_*}, \qquad A \in \mathcal{B}(\mathbb{R}\backslash\{0\}).
\end{aligned} \tag{4.4}$$

The assertion follows from the canonical decomposition of a semimartingale and the triplet of characteristics of the stochastic integral process; see, for example, Proposition 1.3 in Papapantoleon (2007).

Now, using Lemma 2.6 in Kallsen and Shiryaev (2002), we know that we can express the natural exponential of the semimartingale $\overline{Z}$ as the stochastic exponential of another suitable semimartingale Z, i.e. $\mathrm{e}^{\overline{Z}} = \mathcal{E}(Z)$, where Z has dynamics

$$\begin{aligned}
Z &= \overline{Z} + \frac{1}{2}\langle \overline{Z}^c\rangle + (\mathrm{e}^x - 1 - x) * \mu^{\overline{Z}} \\
&= \int_0^{\cdot} \bar{b}(s,T_i)\mathrm{d}s + \int_0^{\cdot} \lambda(s,T_i)\sqrt{c_s}\mathrm{d}W_s^{T_*} + \frac{1}{2}\int_0^{\cdot} \lambda^2(s,T_i)c_s\mathrm{d}s \\
&\quad + \int_0^{\cdot}\int_{\mathbb{R}} \lambda(s,T_i)x(\mu^H - \nu^{T_*})(\mathrm{d}s,\mathrm{d}x) \\
&\quad + \int_0^{\cdot}\int_{\mathbb{R}} (\mathrm{e}^{\lambda(s,T_i)x} - 1 - \lambda(s,T_i)x)\mu^H(\mathrm{d}s,\mathrm{d}x) \\
&= \int_0^{\cdot} \bar{b}(s,T_i)\mathrm{d}s + \frac{1}{2}\int_0^{\cdot} \lambda^2(s,T_i)c_s\mathrm{d}s + \int_0^{\cdot} \lambda(s,T_i)\mathrm{d}H_s \\
&\quad + \int_0^{\cdot}\int_{\mathbb{R}} (\mathrm{e}^{\lambda(s,T_i)x} - 1 - \lambda(s,T_i)x)\mu^H(\mathrm{d}s,\mathrm{d}x).
\end{aligned} \tag{4.5}$$

Therefore, under the terminal measure $\mathbb{P}_{T_*}$ the dynamics of an arbitrary LIBOR rate $L(\cdot,T_i)$ are described by the SDE

$$\mathrm{d}L(t,T_i) = L(t-,T_i)\mathrm{d}Z_t, \tag{4.6}$$

where Z is a semimartingale with decomposition

$$\begin{aligned}
Z &= \int_0^{\cdot} b(s,T_i)\mathrm{d}s + \int_0^{\cdot} \lambda(s,T_i)\mathrm{d}H_s \\
&\quad + \int_0^{\cdot}\int_{\mathbb{R}} \big(\mathrm{e}^{\lambda(s,T_i)x} - 1 - \lambda(s,T_i)x\big)\mu^H(\mathrm{d}s,\mathrm{d}x),
\end{aligned} \tag{4.7}$$

and the (random) drift term, comparing (4.2) and (4.5), is

$$\begin{aligned} b(s,T_i) = -c_s\lambda(s,T_i)\sum_{l=i+1}^{N}\frac{\delta_l L(s-,T_l)}{1+\delta_l L(s-,T_l)}\lambda(s,T_l) \\ -\int_{\mathbb{R}}\left(\left(\mathrm{e}^{\lambda(s,T_i)x}-1\right)\prod_{l=i+1}^{N}\beta(s,x,T_l)-\lambda(s,T_i)x\right)F_s^{T_*}(\mathrm{d}x). \end{aligned} \tag{4.8}$$

The triplet of predictable characteristics of the semimartingale Z under $\mathbb{P}_{T_*}$ is $\mathbb{T}(Z|\mathbb{P}_{T_*}) = (B^Z, C^Z, \nu^Z)$, where

$$\begin{aligned} B^Z &= \int_0^{\cdot}\left(b(s,T_i)+\kappa_s^{ju}(\lambda(s,T_i))\right)\mathrm{d}s \\ C^Z &= \int_0^{\cdot}\lambda^2(s,T_i)c_s\mathrm{d}s \\ 1_A(x)*\nu^Z &= 1_A\left(\mathrm{e}^{\lambda(s,T_i)x}-1\right)*\nu^{T_*}, \qquad A\in\mathcal{B}(\mathbb{R}\backslash\{0\}). \end{aligned} \tag{4.9}$$

Here κ^{ju} denotes the cumulant associated to the *jumps* of the process H, i.e.

$$\kappa_s^{ju}(u) = \int_{\mathbb{R}}(\mathrm{e}^{ux}-1-ux)F_s^{T_*}(\mathrm{d}x). \tag{4.10}$$

Note that the *martingale part* – or equivalently the diffusion and jump characteristics – of the semimartingale Z corresponds to a *Lévy process.*

5. Strong Taylor approximation and applications

The aim of this section is to *strongly approximate* the stochastic differential equations for the dynamics of LIBOR rates under the terminal measure. This pathwise approximation is based on the strong Taylor approximation of the random terms in the drift $b(\cdot,T_i)$ of the semimartingale Z driving the LIBOR rates $L(\cdot,T_i)$, cf. equations (4.6)–(4.8). The essence of the strong Taylor approximation is the perturbation of the initial SDE and a classical Taylor expansion with usual conditions for convergence (cf. Definition 5.1).

We introduce a real parameter ϵ and will approximate the terms

$$L(t-,T_l) \tag{5.1}$$

which cause the drift term to be random, by their *first-order strong Taylor approximation*, cf. Lemma 5.4. Note that the map $x\mapsto\frac{\delta_l x}{1+\delta_l x}$, appearing in the drift, is globally Lipschitz with Lipschitz constant $\delta^*=\max_l\delta_l$.

The following definition of the strong Taylor approximation is taken by Siopacha (2006); see also Siopacha and Teichmann (2007). Consider a smooth curve $\epsilon\mapsto W_\epsilon$, where $\epsilon\in\mathbb{R}$ and $W_\epsilon\in L^2(\Omega;\mathbb{R})$.

Definition 5.1. A *strong Taylor approximation* of order $n\geq 0$ is a (truncated) power series

$$\mathbf{T}^n(W_\epsilon) := \sum_{i=0}^{n}\frac{\epsilon^i}{i!}\frac{\partial^i}{\partial\epsilon^i}\Big|_{\epsilon=0}W_\epsilon \tag{5.2}$$

such that

$$\mathbb{E}\left[|W_\epsilon-\mathbf{T}^n(W_\epsilon)|\right] = o(\epsilon^n), \tag{5.3}$$

holds true as $\epsilon\to 0$.

Then, for Lipschitz functions $f:\mathbb{R}\to\mathbb{R}_{\geqslant 0}$ with Lipschitz constant k we get the following error estimate:

$$\mathbb{E}[|f(W_\epsilon)-f(\mathbf{T}^n(W_\epsilon))|]\leq k\mathbb{E}[|W_\epsilon-\mathbf{T}^n(W_\epsilon)|]=ko(\epsilon^n). \tag{5.4}$$

Remark 5.2. It is important to point out that motivated by the idea of the Taylor series we perform an expansion around $\epsilon=0$ and the estimate (5.4) is valid. However, for the pathwise approximation of LIBOR rates we are interested in the region $\epsilon\approx 1$, and hope that the expansion yields adequate results; for $\epsilon=0$ we would simply recover the "frozen drift" approximation. Numerical experiments show that this approach indeed yields better results than the "frozen drift" approximation; cf. section 6.

We assume that the following condition is in force, which guarantees that $L(t,T_i)\in L^2(\Omega)$, for all $i\in\{0,\dots,N\}$ and all $t\in[0,T_*]$.

Assumption ($\mathbb{L}$). The volatility functions satisfy $|\lambda(s,T_i)|\leq\frac{M}{2}$ for all $s\in[0,T_i]$ and all $i\in\{0,\dots,N\}$.

Let us introduce the process $X^\epsilon(T_i)=(X^\epsilon_t(T_i))_{0\leq t\leq T_*}$ with initial values $X^\epsilon_0(T_i)=L(0,T_i)$ for all $i\in\{0,\dots,N\}$ and all $\epsilon\in\mathbb{R}$. The dynamics of $X^\epsilon(T_i)$ is described by perturbing the SDE of the LIBOR rates by the perturbation parameter ϵ:

$$\begin{aligned}\mathrm{d}X^\epsilon_t(T_i)=\epsilon X^\epsilon_{t-}(T_i)\Big(&b^\epsilon(t,T_i)\mathrm{d}t+\lambda(t,T_i)\mathrm{d}H_t\\&+\int_{\mathbb{R}}\left(\mathrm{e}^{\lambda(t,T_i)x}-1-\lambda(t,T_i)x\right)\mu^H(\mathrm{d}t,\mathrm{d}x)\Big),\end{aligned} \tag{5.5}$$

where,

$$\begin{aligned}b^\epsilon(t,T_i)=&-c_t\lambda(t,T_i)\sum_{l=i+1}^{N}\frac{\delta_l X^\epsilon_{t-}(T_l)}{1+\delta_l X^\epsilon_{t-}(T_l)}\lambda(t,T_l)\\&-\int_{\mathbb{R}}\left(\left(\mathrm{e}^{\lambda(t,T_i)x}-1\right)\prod_{l=i+1}^{N}\beta^\epsilon(t,x,T_l)-\lambda(t,T_i)x\right)F^{T_*}_t(\mathrm{d}x),\end{aligned} \tag{5.6}$$

and

$$\beta^\epsilon(t,x,T_l)=\frac{\delta_l X^\epsilon_{t-}(T_l)}{1+\delta_l X^\epsilon_{t-}(T_l)}\left(\mathrm{e}^{\lambda(t,T_l)x}-1\right)+1. \tag{5.7}$$

Remark 5.3. In the sequel we will use the notation $\mathbf{T}$ as shorthand for $\mathbf{T}^1$.

Lemma 5.4. *The* first-order strong Taylor approximation *of the random variable $X^\epsilon_t(T_i)$ is given by:*

$$\mathbf{T}\big(X^\epsilon_t(T_i)\big)=L(0,T_i)+\epsilon\frac{\partial}{\partial\epsilon}\big|_{\epsilon=0}X^\epsilon_t(T_i), \tag{5.8}$$

where the first variation process of $X^\epsilon(T_i)$, denoted by $Y(T_i) := \frac{\partial}{\partial\epsilon}|_{\epsilon=0}X^\epsilon(T_i)$, is a time-inhomogeneous Lévy process with local characteristics

$$\begin{aligned} b_s^{Y_i} &= L(0,T_i)\Big(b^0(s,T_i) + \kappa_s^{ju}(\lambda(s,T_i))\Big) \\ c_s^{Y_i} &= \Big(L(0,T_i)\lambda(s,T_i)\Big)^2 c_s \end{aligned} \tag{5.9}$$

$$\int 1_A(x)F_s^{Y_i}(\mathrm{d}x) = \int 1_A\Big(L(0,T_i)(\mathrm{e}^{\lambda(s,T_i)x}-1)\Big)F^{T_*}(\mathrm{d}x), \quad A \in \mathcal{B}(\mathbb{R}).$$

Proof. By definition, the first-order strong Taylor approximation is given by the truncated power series

$$\mathbf{T}\big(X_t^\epsilon(T_i)\big) = X_t^0(T_i) + \epsilon\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}X_t^\epsilon(T_i). \tag{5.10}$$

Since the curves $\epsilon \mapsto X_t^\epsilon(T_i)$ are smooth, and $X_t^\epsilon(T_i) \in L^2(\Omega)$ by Assumption ($\mathbb{L}$), we get that strong Taylor approximations of arbitrary order can always be obtained, cf. Kriegl and Michor (1997, Chapter 1). In particular, for the first-order expansion we have that

$$\mathbb{E}\left[|X_t^\epsilon(T_i) - \mathbf{T}\big(X_t^\epsilon(T_i)\big)|\right] = o(\epsilon). \tag{5.11}$$

The zero-order term of the Taylor expansion trivially is

$$X_t^0(T_i) = X_0^0(T_i) \text{ for all } t, \text{ since } \mathrm{d}X_t^0(T_i) = 0.$$

Of course, the initial values of the perturbed SDE coincide with the initial values of the un-perturbed SDE, hence $X_0^0(T_i) = L(0,T_i)$.

The first variation process of $X^\epsilon(T_i)$ with respect to ϵ is derived by differentiating (5.5); hence, the dynamics is

$$\begin{aligned} \mathrm{d}\Big(\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}X_t^\epsilon(T_i)\Big) &= X_{t-}^\epsilon(T_i)|_{\epsilon=0}\Big(b^\epsilon(t,T_i)|_{\epsilon=0}\mathrm{d}t + \lambda(t,T_i)\mathrm{d}H_t \\ &\qquad + \int_{\mathbb{R}} \big(\mathrm{e}^{\lambda(t,T_i)x} - 1 - \lambda(t,T_i)x\big)\mu^H(\mathrm{d}t,\mathrm{d}x)\Big) \\ &= X_0^0(T_i)\Big(b^0(t,T_i)\mathrm{d}t + \lambda(t,T_i)\mathrm{d}H_t \\ &\qquad + \int_{\mathbb{R}} \big(\mathrm{e}^{\lambda(t,T_i)x} - 1 - \lambda(t,T_i)x\big)\mu^H(\mathrm{d}t,\mathrm{d}x)\Big). \end{aligned} \tag{5.12}$$

We can immediately notice that in the drift term $b^0(t,T_i)$ of the first variation process, the random terms $X_t^\epsilon(T_i)$ are replaced by their deterministic initial values $X_0^0(T_i) = L(0,T_i)$.

Let us denote by $Y(T_i)$ the first variation process of $X^\epsilon(T_i)$, i.e. $Y(T_i) = \frac{\partial}{\partial\epsilon}|_{\epsilon=0}X^\epsilon(T_i)$. The solution of the linear SDE (5.12) describing the dynamics of the first variation process yields

$$\begin{aligned} Y_t(T_i) = L(0,T_i)\Big(&\int_0^t b^0(s,T_i)\mathrm{d}s + \int_0^t \lambda(s,T_i)\mathrm{d}H_s \\ &+ \int_0^t\int_{\mathbb{R}} \big(\mathrm{e}^{\lambda(s,T_i)x} - 1 - \lambda(s,T_i)x\big)\mu^H(\mathrm{d}s,\mathrm{d}x)\Big). \end{aligned} \tag{5.13}$$

Since the drift term is deterministic, H is a time-inhomogeneous Lévy process and μ^H is the random measure of jumps of a time-inhomogeneous Lévy process, we can conclude that Y is itself a time-inhomogeneous Lévy process. The local characteristics of Y are described by (5.9). □

Concludingly, we have the following strong Taylor approximation scheme for the random terms (5.1) entering the drift:

$$X_t^\epsilon(T_i) = L(0,T_i) + \epsilon Y_t(T_i) + o(\epsilon). \tag{5.14}$$

Remark 5.5. A consequence of this approximation scheme is that we can interpret the "frozen drift" approximation – first proposed by Brace et al. (1997) and then used by several other authors – as the *zero-order* Taylor approximation, i.e. $X_t^\epsilon(T_i) \approx L(0,T_i)$.

Remark 5.6. Let us elaborate on the random terms of the first variation process $Y(T_i)$. Define the process

$$\overline{U} = \int_0^{\cdot} \lambda(s,T_i)\mathrm{d}H_s,$$

and consider the exponential transform of $\overline{U}$, i.e. the process U such that $U := \mathcal{L}\mathrm{og}(\mathrm{e}^{\overline{U}})$. Applying Lemma 2.6 in Kallsen and Shiryaev (2002) we know that U and $\overline{U}$ are related via

$$U = \overline{U} + \frac{1}{2}\langle \overline{U}^c, \overline{U}^c\rangle + (\mathrm{e}^x - 1 - x) * \mu^{\overline{U}}. \tag{5.15}$$

We immediately observe that up to the multiplicative constant $L(0,T_i)$, the martingale parts of U and of the first variation process $Y(T_i)$ (5.13) are indistinguishable. Now, by the definition of the stochastic logarithm we know that U is the solution of the following stochastic differential equation:

$$\mathrm{d}U = \frac{1}{\mathrm{e}^{\overline{U}_-}}\mathrm{d}\big(\mathrm{e}^{\overline{U}}\big) \Longleftrightarrow U = \int \frac{1}{\mathrm{e}^{\overline{U}_-}}\mathrm{d}\big(\mathrm{e}^{\overline{U}}\big). \tag{5.16}$$

The latter observation is useful for the discretization and simulation of the first variation process.

Next, we parameterize the Lévy LIBOR model in terms of the perturbation parameter ϵ; that is, the stochastic differential equation describing the evolution of the LIBOR rates takes the form

$$\begin{aligned}\mathrm{d}L^\epsilon(t,T_i) = L^\epsilon(t-,T_i)\Big(& b^\epsilon(t,T_i)\mathrm{d}t + \lambda(t,T_i)\mathrm{d}H_t \\ &+ \int_{\mathbb{R}} \big(\mathrm{e}^{\lambda(t,T_i)x} - 1 - \lambda(t,T_i)x\big)\mu^H(\mathrm{d}t,\mathrm{d}x)\Big),\end{aligned} \tag{5.17}$$

where $b^\epsilon(t,T_i)$ and $\beta^\epsilon(t,x,T_i)$ are given by (5.6) and (5.7) respectively, hence they involve the process $X^\epsilon(T_i)$. Moreover, let $L^\epsilon(0,T_i) = L(0,T_i)$ for all values of $\epsilon \in \mathbb{R}$ and all LIBOR rates, i.e. for all $i \in \{0,\dots,N\}$. Notice that for $\epsilon = 1$ the processes $X^\epsilon(T_i)$, $L^\epsilon(\cdot,T_i)$ and $L(\cdot,T_i)$ are *indistinguishable.*

Now, we provide a way to pathwise approximate $L^\epsilon(\cdot,T_i)$ by means of the strong Taylor approximation.

Proposition 5.7. *Assume that LIBOR rates are modeled according to the Lévy LIBOR model and let $\epsilon \in \mathbb{R}$. Then, the stochastic differential equation for $L^\epsilon(\cdot, T_i)$, given by (5.17), (5.6) and (5.7), can be strongly approximated as $\epsilon \downarrow 0$ by*

$$\begin{aligned} \mathrm{d}\widehat{L}^\epsilon(t,T_i) = \widehat{L}^\epsilon(t-,T_i)\Big(&\widehat{b}^\epsilon(t,T_i)\mathrm{d}t + \lambda(t,T_i)\mathrm{d}H_t \\ &+ \int_{\mathbb{R}} \big(\mathrm{e}^{\lambda(t,T_i)x} - 1 - \lambda(t,T_i)x\big)\mu^H(\mathrm{d}t,\mathrm{d}x)\Big), \end{aligned} \tag{5.18}$$

where

$$\begin{aligned} \widehat{b}^\epsilon(t,T_i) = &-c_t\lambda(t,T_i)\sum_{l=i+1}^{N} \frac{\delta_l \mathbf{T}\big(X^\epsilon_{t-}(T_l)\big)_+}{1+\delta_l \mathbf{T}\big(X^\epsilon_{t-}(T_l)\big)_+}\lambda(t,T_l) \\ &- \int_{\mathbb{R}} \left(\Big(\mathrm{e}^{\lambda(t,T_i)x} - 1\Big)\prod_{l=i+1}^{N}\widehat{\beta}^\epsilon(t,x,T_l) - \lambda(t,T_i)x\right)F_t^{T_*}(\mathrm{d}x), \end{aligned} \tag{5.19}$$

and

$$\widehat{\beta}^\epsilon(t,x,T_l) = \frac{\delta_l \mathbf{T}\big(X^\epsilon_{t-}(T_l)\big)_+}{1+\delta_l \mathbf{T}\big(X^\epsilon_{t-}(T_l)\big)_+}\Big(\mathrm{e}^{\lambda(t,T_l)x} - 1\Big) + 1. \tag{5.20}$$

Proof. The first step is to interchange $X^\epsilon(T_i)$ with $X^\epsilon(T_i)_+$, which yields no difference to the dynamics of $L^\epsilon(\cdot,T_i)$ in (5.17), (5.6) and (5.7), since $X^\epsilon_t(T_i)$ is a positive random variable. Then, using Taylor's expansion, we know that $\widehat{L}^\epsilon(t,T_i) \to L^\epsilon(t,T_i)$ $\mathbb{P}$-a.s as $\epsilon \downarrow 0$.

The error in the approximation stems from the drift term, since the martingale parts of $L^\epsilon(\cdot,T_i)$ and $\widehat{L}^\epsilon(t,T_i)$ are identical. We have the following estimate for the error term:

$$\begin{aligned} &\big|\log \widehat{L}^\epsilon(t,T_i) - \log L^\epsilon(t,T_i)\big| \\ &\quad\le \int_0^t \big|\widehat{b}^\epsilon(s,T_i) - b^\epsilon(s,T_i)\big|\mathrm{d}s \\ &\quad\le \int_0^t \Big|c_s\lambda(s,T_i)\sum_{l=i+1}^{N}\Big(\alpha^\epsilon(s,T_l) - \widehat{\alpha}^\epsilon(s,T_l)\Big)\Big|\mathrm{d}s \\ &\quad+ \int_0^t \Big|\int_{\mathbb{R}}\Big(\mathrm{e}^{\lambda(s,T_i)x} - 1\Big)\left(\prod_{l=i+1}^{N}\beta^\epsilon(s,x,T_l) - \prod_{l=i+1}^{N}\widehat{\beta}^\epsilon(s,x,T_l)\right)F_s^{T_*}(\mathrm{d}x)\Big|\mathrm{d}s. \end{aligned}$$

Now, for the first term we can estimate using the Lipschitz property of $x \mapsto \frac{\delta_l x}{1+\delta_l x}$ and Assumption ($\mathbb{L}$):

$$\int_0^t \Big| c_s \lambda(s,T_i) \sum_{l=i+1}^N \bigg(\frac{\delta_l X^\epsilon_{s-}(T_l)}{1+\delta_l X^\epsilon_{s-}(T_l)} - \frac{\delta_l \mathbf{T}\big(X^\epsilon_{s-}(T_l)\big)_+}{1+\delta_l \mathbf{T}\big(X^\epsilon_{s-}(T_l)\big)_+} \bigg) \lambda(s,T_l) \Big| \mathrm{d}s$$
$$\leq \int_0^t c_s |\lambda(s,T_i)| \delta^* \sum_{l=i+1}^N \Big| X^\epsilon_{s-}(T_l) - \mathbf{T}\big(X^\epsilon_{s-}(T_l)\big)_+ \Big| |\lambda(s,T_l)| \mathrm{d}s$$
$$\leq \int_0^t c_s (N-i-1) \frac{M^2}{4} \delta^* \sum_{l=i+1}^N \Big| X^\epsilon_{s-}(T_l) - \mathbf{T}\big(X^\epsilon_{s-}(T_l)\big)_+ \Big| \mathrm{d}s.$$

Similarly for the second term, using again the Lipschitz property, Assumptions ($\mathbb{EM}$) and ($\mathbb{L}$), and in addition that $\prod_{k=1}^n (a_k+1) \simeq 1 + \sum_{k=1}^n a_k$ for small a_k, we get

$$\int_0^t \Big| \int_{\mathbb{R}} \left(\mathrm{e}^{\lambda(s,T_i)x} - 1 \right) \left(\prod_{l=i+1}^N \beta^\epsilon(s,x,T_l) - \prod_{l=i+1}^N \widehat{\beta}^\epsilon(s,x,T_l) \right) F_s^{T_*}(\mathrm{d}x) \Big| \mathrm{d}s$$
$$\leq \int_0^t \int_{\mathbb{R}} \Big| \mathrm{e}^{\lambda(s,T_i)x} - 1 \Big| \delta^* \sum_{l=i+1}^N \Big| X^\epsilon_{s-}(T_l) - \mathbf{T}\big(X^\epsilon_{s-}(T_l)\big)_+ \Big| \Big| \mathrm{e}^{\lambda(s,T_l)x} - 1 \Big| F_s^{T_*}(\mathrm{d}x) \mathrm{d}s$$
$$\leq \int_0^t \kappa(M)(N-i-1) \delta^* \sum_{l=i+1}^N \Big| X^\epsilon_{s-}(T_l) - \mathbf{T}\big(X^\epsilon_{s-}(T_l)\big)_+ \Big| \mathrm{d}s.$$

Therefore we can conclude that

$$\big| \log \widehat{L}^\epsilon(t,T_i) - \log L^\epsilon(t,T_i) \big| \leq \int_0^t \mathcal{C}_s \, \delta^* \sum_{l=i+1}^N \Big| X^\epsilon_{s-}(T_l) - \mathbf{T}\big(X^\epsilon_{s-}(T_l)\big)_+ \Big| \mathrm{d}s,$$

where $\mathcal{C}_s := \mathcal{C}_s(N,M)$ is a constant depending on s, N and M. □

Naturally, we can solve the stochastic differential equation (5.18) describing the dynamics of the approximate LIBOR rate $\widehat{L}^\epsilon(\cdot,T_i)$. Let $\widehat{Z}^\epsilon$ denote the semimartingale

$$\widehat{Z}^\epsilon = \int_0^\cdot \widehat{b}^\epsilon(t,T_i)\mathrm{d}t + \int_0^\cdot \lambda(t,T_i)\mathrm{d}H_t$$
$$+ \int_0^\cdot \int_{\mathbb{R}} \big(\mathrm{e}^{\lambda(t,T_i)x} - 1 - \lambda(t,T_i)x \big) \mu^H(\mathrm{d}t,\mathrm{d}x);$$

T	0.5 Y	1 Y	1.5 Y	2 Y	2.5 Y
$B(0,T)$	0.9833630	0.9647388	0.9435826	0.9228903	0.9006922
T	3 Y	3.5 Y	4 Y	4.5 Y	5 Y
$B(0,T)$	0.8790279	0.8568412	0.8352144	0.8133497	0.7920573

TABLE 6.1. Euro zero coupon bond prices on February 19, 2002.

then we get that

$$\begin{aligned}
\mathrm{d}\widehat{L}^\epsilon(t,T_i) = \widehat{L}^\epsilon(t-,T_i)\mathrm{d}\widehat{Z}^\epsilon_t &\iff \\
\widehat{L}^\epsilon(t,T_i) &= \widehat{L}^\epsilon(0,T_i)\mathcal{E}(\widehat{Z}^\epsilon)_t \\
&= \widehat{L}^\epsilon(0,T_i)\exp\Big(\widehat{Z}^\epsilon_t - \frac{1}{2}\langle \widehat{Z}^{\epsilon,c}, \widehat{Z}^{\epsilon,c}\rangle_t + \big(\log(1+x)-x\big) * \mu^{\widehat{Z}^\epsilon}_t\Big) \\
&= \widehat{L}^\epsilon(0,T_i)\exp\Big(\int_0^t \overline{b}^\epsilon(s,T_i)\mathrm{d}s + \int_0^t \lambda(s,T_i)\mathrm{d}H_s\Big), \qquad (5.21)
\end{aligned}$$

where

$$\overline{b}^\epsilon(s,T_i) = \widehat{b}^\epsilon(s,T_i) - \frac{1}{2}\lambda^2(s,T_i)c_s, \qquad (5.22)$$

and $\widehat{b}^\epsilon(s,T_i)$ is given by (5.19)–(5.20).

Remark 5.8. Let us point out again that this method can be applied to any LIBOR model driven by a general semimartingale; indeed, the properties of Lévy processes are not *essential* in the proofs of Lemma 5.4 or Proposition 5.7. If we start with a LIBOR model driven by a general semimartingale then the structure of this semimartingale will be "transferred" to the first variation process, and hence to the dynamics of the strong Taylor approximation.

6. Numerical illustration

The aim of this section is to demonstrate the accuracy and efficiency of the Taylor approximation scheme for the valuation of options in the Lévy LIBOR model compared to the "frozen drift" approximation. We will consider the pricing of caps and swaptions, although other interest rate derivatives can be considered in this framework.

We revisit the numerical example in Kluge (2005, pp. 76-83). That is, we consider a tenor structure $T_0 = 0, T_1 = \frac{1}{2}, T_2 = 1\dots, T_{10} = 5 = T_*$, constant volatilities

$$\begin{aligned}
\lambda(s,T_1) &= 0.20 & \lambda(s,T_2) &= 0.19 & \lambda(s,T_3) &= 0.18 \\
\lambda(s,T_4) &= 0.17 & \lambda(s,T_5) &= 0.16 & \lambda(s,T_6) &= 0.15 \\
\lambda(s,T_7) &= 0.14 & \lambda(s,T_8) &= 0.13 & \lambda(s,T_9) &= 0.12
\end{aligned}$$

and the discount factors (zero coupon bond prices) as quoted on February 19, 2002; cf. Table 6.1. The tenor length is constant and denoted by $\delta = \frac{1}{2}$.

The driving Lévy process H is a normal inverse Gaussian (NIG) process with parameters $\alpha = \bar{\delta} = 1.5$ and $\mu = \beta = 0$. We denote by μ^H the random

measure of jumps of H and by $\nu(\mathrm{d}t, \mathrm{d}x) = F(\mathrm{d}x)\mathrm{d}t$ the $\mathbb{P}_{T_*}$-compensator of μ^H, where F is the Lévy measure of the NIG process. The necessary conditions are satisfied because $M = \alpha$, hence $\sum_{i=1}^{9} |\lambda(s, T_i)| = 1.44 < \alpha$ and $\lambda(s, T_i) < \frac{\alpha}{2}$, for all $i \in \{1, \ldots, 9\}$.

The NIG Lévy process is a pure-jump Lévy process and, for $\mu = 0$, has the canonical decomposition

$$H = \int_0^{\cdot} \int_{\mathbb{R}} x(\mu^H - \nu)(\mathrm{d}s, \mathrm{d}x). \tag{6.1}$$

The characteristic function of the NIG distribution is

$$\varphi_{H_1}(u) = \exp\left(\bar{\delta}\alpha - \bar{\delta}\sqrt{\alpha^2 + u^2}\right) \tag{6.2}$$

for all $u \in \mathbb{R}$; moreover, from (2.6), (2.7) and (6.2), we get that

$$\begin{aligned} \kappa(u) &= \int_{\mathbb{R}} (\mathrm{e}^{ux} - 1 - ux) F(\mathrm{d}x) = \log \mathbb{E}[\exp(uH_1)] \\ &= \bar{\delta}\alpha - \bar{\delta}\sqrt{\alpha^2 - u^2}, \end{aligned} \tag{6.3}$$

for all $u \in \mathbb{C}$ with $|\Re u| \leq \alpha$.

6.1. **Caplets.** The price of a caplet with strike K maturing at time T_i, using the relationship between the terminal and the forward measures cf. (3.6), can be expressed as

$$\begin{aligned} C_0(K, T_i) &= \delta B(0, T_{i+1})\, \mathbb{E}_{\mathbb{P}_{T_{i+1}}}[(L(T_i, T_i) - K)^+] \\ &= \delta B(0, T_{i+1})\, \mathbb{E}_{\mathbb{P}_{T_*}}\Big[\frac{\mathrm{d}\mathbb{P}_{T_{i+1}}}{\mathrm{d}\mathbb{P}_{T_*}}\big|_{\mathcal{F}_{T_i}} (L(T_i, T_i) - K)^+\Big] \\ &= \delta B(0, T_*)\, \mathbb{E}_{\mathbb{P}_{T_*}}\Big[\prod_{l=i+1}^{N} \big(1 + \delta L(T_i, T_l)\big)(L(T_i, T_i) - K)^+\Big]. \end{aligned} \tag{6.4}$$

This equation will provide the *actual* prices of caplets corresponding to simulating the full SDE for the LIBOR rates. In order to calculate the first-order Taylor approximation prices for a caplet we have to replace $L(\cdot, T_\cdot)$ in (6.4) with $\widehat{L}^{\epsilon}(\cdot, T_\cdot)$. Similarly, for the frozen drift approximation prices we must use $\widehat{L}^{0}(\cdot, T_\cdot)$ instead of $L(\cdot, T_\cdot)$.

The main difficulty when simulating the full SDE for the dynamics of LIBOR rates is that each LIBOR rate depends on *all* previous LIBOR rates, see equations (3.7)–(3.10) or (4.6)–(4.8). Hence, we are facing a system of *highly nested* SDEs that need to be simulated simultaneously. Indeed, the "dependence" structure of LIBOR rates takes the form of a (lower) triangular matrix shown in Table 6.2.

The strong Taylor approximation method navigates around this problem, since each approximate LIBOR rate depends on all the Taylor approximations of the previous LIBOR rates, which are *independent* of each other. Hence, the resulting SDEs are *not nested* and can be simulated in parallel, see equations (5.18)–(5.20) and (5.8)–(5.9). In other words, we can simulate $\widehat{L}^{\epsilon}(\cdot, T_i)$, *without* having to simulate all previous $N - (i + 1)$ approximate LIBOR rates. This also means that the rate of convergence of numerical

$\ddots$						
$\ddots$	$L(t,T_{i+1})$					
$\ddots$		$\ddots$				
	$L(t,T_{N-2})$	$\dots$	$L(t,T_{N-2})$			
	$L(t,T_{N-1})$	$\dots$	$L(t,T_{N-1})$	$L(t,T_{N-1})$		
$\dots$	$L(t,T_N)$	$\dots$	$L(t,T_N)$	$L(t,T_N)$	$L(t,T_N)$	
$\dots$	$L(t,T_i)$	$\dots$	$L(t,T_{N-3})$	$L(t,T_{N-2})$	$L(t,T_{N-1})$	$L(t,T_N)$

TABLE 6.2. Matrix of dependencies for LIBOR rates

schemes for $\widehat{L}^\epsilon(\cdot,T_i)$ is usually higher than for the original system. The numerical experiments show that the Taylor approximation method reduces the computational complexity of the model, while maintaining the accuracy of prices.

We will compare the performance of the strong Taylor approximation relative to the frozen drift approximation in two ways: (1) we compute the prices of caplets by simulating separately the full SDE, the strong Taylor approximation SDE and the frozen drift SDE and compare the prices in terms of their implied volatilities; (2) in order to get a better picture of the performance of the two methods, we simulate the *difference* between the full SDE and the strong Taylor or the frozen drift prices; this resembles a variance reduction technique in Monte Carlo methods and results in a very low standard error of the Monte Carlo simulations.

The implied volatility surface corresponding to the full SDE prices can be seen in Table A.1. In Figure 6.1 we present the difference in implied volatility between the full SDE prices and the frozen drift prices, while in Figure 6.2 we show the difference in implied volatility between the full SDE prices and the strong Taylor prices. One can immediately observe that the strong Taylor approximation method performs much better than the frozen drift approximation; the difference in implied volatilities is very low across all strikes and maturities. Indeed, the difference in implied volatility between the full SDE and the strong Taylor prices lies always below the 1% threshold, which deems this approximation accurate enough for practical implementations. On the contrary, the difference in implied volatilities for the frozen drift approximation exceeds the 1% level for in-the-money options.

The implied volatilities for all three methods and the standard error of the Monte Carlo simulation for the full SDE prices are reported in Appendix A. Note that we have used 1 million paths and 200 time steps in all the Monte Carlo simulations.

However, the standard error in these Monte Carlo simulations is relatively high, cf. Table B.4. In order to reduce this error and get a more accurate picture of the performance of the methods we simulate the *differences* between caplet prices in the different methods and get numerical results with relatively low Monte Carlo error. Figure 6.3 exhibits the difference between the strong Taylor approximation caplet prices and the full SDE prices, while Figure 6.4 exhibits the difference between the frozen drift and the full SDE

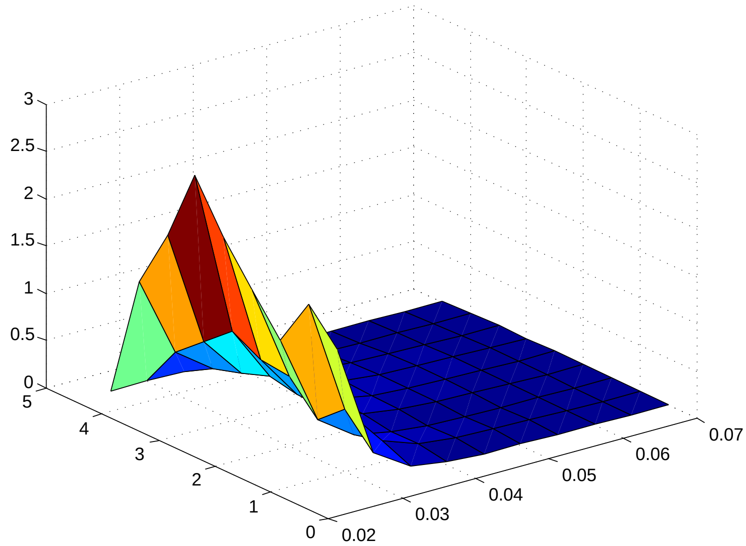

FIGURE 6.1. Difference in implied volatilities between the full SDE and the frozen drift prices.

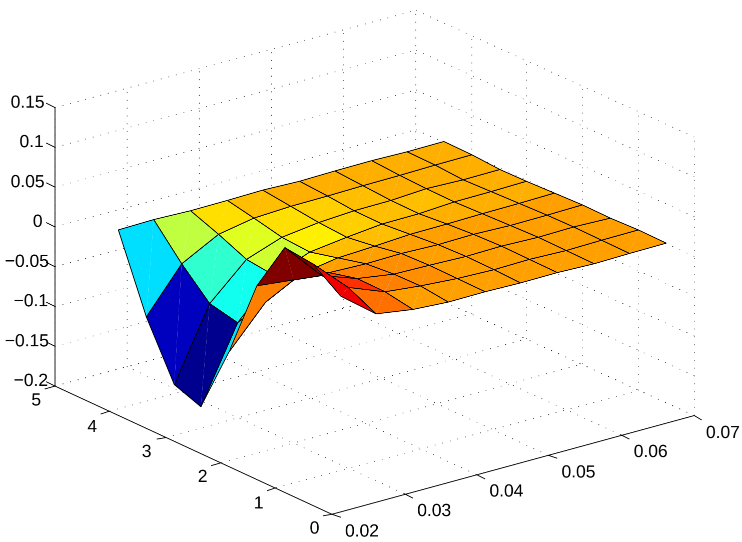

FIGURE 6.2. Difference in implied volatilities between the full SDE and the strong Taylor prices.

caplet prices. The respective standard errors for the Monte Carlo simulation are reported in Tables A.5 and A.6. We can immediately observe that the difference between the strong Taylor and the full SDE prices is significantly smaller than the difference between the frozen drift and the full SDE prices. Moreover, the respective standard error is several orders of magnitude smaller, pointing again to the superior performance of the strong Taylor approximation relative to the frozen drift approximation.

6.2. **Swaptions.** Next, we will consider the pricing of swaptions. Recall that a payer (resp. receiver) swaption can be viewed as a put (resp. call)

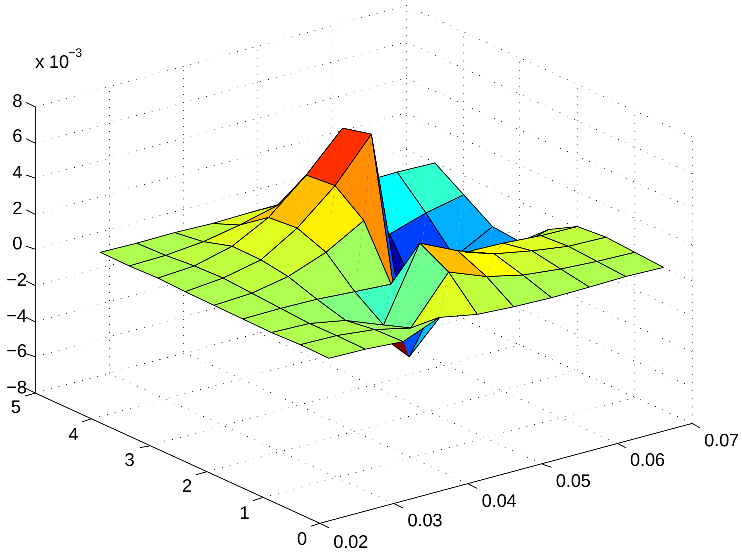

FIGURE 6.3. Difference in caplet prices between the strong Taylor and full SDE method.

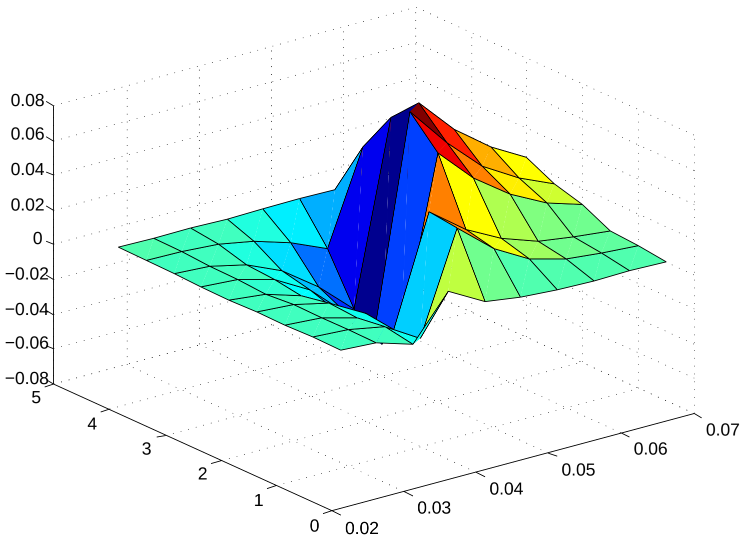

FIGURE 6.4. Difference in caplet prices between the frozen drift and full SDE method.

option on a coupon bond with exercise price 1; cf. section 16.2.3 and 16.3.2 in Musiela and Rutkowski (1997). Consider a payer swaption with strike rate K, where the underlying swap starts at time T_i and matures at T_m

$(i < m \leq N)$. The time-T_i value is

$$\begin{aligned}\mathbb{S}_{T_i}(K, T_i, T_m) &= \left(1 - \sum_{k=i+1}^{m} c_k B(T_i, T_k)\right)^+ \\ &= \left(1 - \sum_{k=i+1}^{m} \left(c_k \prod_{l=i}^{k-1} \frac{1}{1 + \delta L(T_i, T_l)}\right)\right)^+, \qquad (6.5)\end{aligned}$$

where

$$c_k = \begin{cases} K, & i+1 \leq k \leq m-1, \\ 1+K, & k = m. \end{cases} \qquad (6.6)$$

Then, the time-0 value of the swaption is obtained by taking the $\mathbb{P}_{T_i}$-expectation of its time-T_i value, hence

$$\begin{aligned}\mathbb{S}_0 &= \mathbb{S}_0(K, T_i, T_m) \\ &= B(0, T_i)\, \mathbb{E}_{\mathbf{P}_{T_i}} \left[\left(1 - \sum_{k=i+1}^{m} \left(c_k \prod_{l=i}^{k-1} \frac{1}{1 + \delta L(T_i, T_l)}\right)\right)^+\right] \\ &= B(0, T_*) \\ &\quad \times \mathbb{E}_{\mathbf{P}_{T_*}} \left[\prod_{l=i}^{N} (1 + \delta L(T_i, T_l)) \left(1 - \sum_{k=i+1}^{m} \left(c_k \prod_{l=i}^{k-1} \frac{1}{1 + \delta L(T_i, T_l)}\right)\right)^+\right] \\ &= B(0, T_*)\, \mathbb{E}_{\mathbf{P}_{T_*}} \left[\left(- \sum_{k=i}^{m} \left(c_k \prod_{l=k}^{N} (1 + \delta L(T_i, T_l))\right)\right)^+\right], \qquad (6.7)\end{aligned}$$

where $c_i := -1$. Once again, this equation will provide the *actual* prices of swaptions corresponding to simulating the full SDE for the LIBOR rates. In order to calculate the first-order Taylor approximation prices we have to replace $L(\cdot, T_\cdot)$ with $\widehat{L}^\epsilon(\cdot, T_\cdot)$, and for the frozen drift approximation prices we must use $\widehat{L}^0(\cdot, T_\cdot)$ instead of $L(\cdot, T_\cdot)$.

We will price eight swaptions in our tenor structure; we consider 1 year and 2 years as option maturities, and then use 12, 18, 24 and 30 months as swap maturities for each option. Similarly to the simulations we performed for caplets, we first simulate the *prices* of swaptions using all three methods, and then simulate the *differences* in prices of swaptions between the full SDE and the strong Taylor method, and the full SDE and the frozen drift method, in order to reduce the standard error of the Monte Carlo simulation.

The prices of swaptions corresponding to the simulation of the full SDE for the LIBOR rates are reported in Table B.1, while the prices corresponding to the strong Taylor method and the frozen drift method are reported in Tables B.2 and B.3 respectively. Once again we observe that the strong Taylor method is performing very well across all strikes, option maturities and swap maturities, while the performance of the frozen drift method is poor for in-the-money swaptions and seems to be deteriorating for longer swap maturities. This observation is in accordance with the common knowledge that the frozen drift approximation is performing worse and worse for longer maturities.

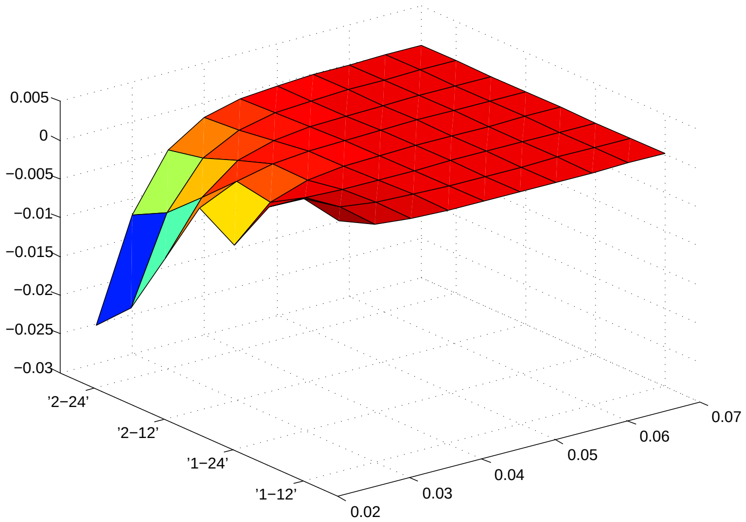

FIGURE 6.5. Difference in swaption prices between the strong Taylor and full SDE method.

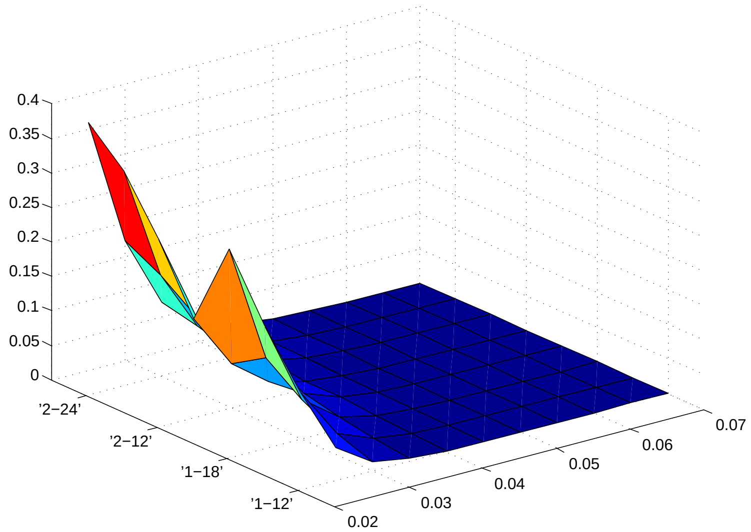

FIGURE 6.6. Difference in swaption prices between the frozen drift and full SDE method.

Next, we have simulated the differences in swaption prices between the full SDE and the strong Taylor method and the full SDE and the frozen drift method; the differences in swaption prices are shown in Figures 6.5 and 6.6 respectively. The standard errors of the corresponding Monte Carlo simulations are reported in Tables B.5 and B.6 respectively. The performance of the strong Taylor method is clearly superior in comparison to the frozen drift method, while we can observe once again that the standard error of the full SDE-strong Taylor simulation is significantly smaller than the standard error of the full SDE-frozen drift simulation.

Appendix A. Caplet prices, standard errors and implied Black volatilities.

In this appendix we report the implied Black volatilities and the standard errors of the Monte Carlo simulation for the caplet prices corresponding to the full SDE for the LIBOR rates. We also report the implied volatilities of the strong Taylor approximation and the frozen drift approximation caplet prices. The standard errors in these cases are omitted for the sake of brevity; we point out though that the standard errors are very similar in all three methods.

	2.5 %	3.0 %	3.5 %	4.0 %	4.5 %	5.0 %	5.5 %	6.0 %	6.5 %	7.0 %
T_1	27.27	22.11	19.09	18.88	20.93	23.35	25.59	27.59	29.36	30.88
T_2	26.09	21.71	19.53	18.49	18.36	18.87	19.73	20.71	21.67	22.59
T_3	21.90	19.31	18.12	17.62	17.58	17.88	18.35	18.92	19.51	20.09
T_4	22.22	19.07	17.65	17.01	16.77	16.77	16.94	17.21	17.55	17.91
T_5	19.84	17.42	16.42	15.99	15.82	15.83	15.94	16.14	16.38	16.64
T_6	18.82	16.47	15.51	15.09	14.90	14.85	14.89	15.01	15.17	15.34
T_7	17.24	15.22	14.41	14.07	13.91	13.88	13.92	14.01	14.14	14.28
T_8	15.75	14.03	13.36	13.07	12.93	12.88	12.90	12.95	13.05	13.15
T_9	11.90	12.27	12.10	11.96	11.88	11.85	11.87	11.93	12.01	12.10

Table A.1. Implied Black volatilities for full SDE prices.

	2.5 %	3.0 %	3.5 %	4.0 %	4.5 %	5.0 %	5.5 %	6.0 %	6.5 %	7.0 %
T_1	27.23	22.10	19.09	18.88	20.93	23.35	25.59	27.59	29.36	30.88
T_2	26.02	21.68	19.52	18.48	18.36	18.87	19.73	20.71	21.67	22.59
T_3	21.82	19.28	18.11	17.61	17.58	17.87	18.35	18.92	19.51	20.09
T_4	22.21	19.06	17.65	17.01	16.77	16.77	16.94	17.21	17.55	17.91
T_5	19.93	17.46	16.44	16.00	15.83	15.83	15.95	16.14	16.38	16.64
T_6	19.00	16.56	15.54	15.10	14.91	14.86	14.90	15.01	15.17	15.34
T_7	17.41	15.29	14.44	14.08	13.92	13.88	13.92	14.01	14.14	14.28
T_8	15.84	14.07	13.38	13.08	12.94	12.89	12.90	12.96	13.05	13.15
T_9	11.90	12.27	12.10	11.96	11.88	11.85	11.87	11.93	12.01	12.10

Table A.2. Implied Black volatilities for strong Taylor prices.

	2.5 %	3.0 %	3.5 %	4.0 %	4.5 %	5.0 %	5.5 %	6.0 %	6.5 %	7.0 %
T_1	25.74	21.76	19.00	18.85	20.91	23.34	25.58	27.59	29.35	30.88
T_2	24.21	21.04	19.31	18.40	18.31	18.84	19.71	20.69	21.66	22.58
T_3	20.54	18.89	17.97	17.55	17.54	17.85	18.33	18.90	19.50	20.08
T_4	20.45	18.33	17.36	16.87	16.69	16.72	16.90	17.19	17.52	17.89
T_5	17.65	16.64	16.12	15.84	15.73	15.77	15.90	16.10	16.35	16.62
T_6	16.13	15.53	15.15	14.91	14.80	14.78	14.85	14.97	15.14	15.32
T_7	15.32	14.54	14.15	13.94	13.84	13.83	13.89	13.98	14.11	14.26
T_8	14.45	13.59	13.21	13.00	12.89	12.86	12.88	12.94	13.03	13.14
T_9	11.90	12.27	12.10	11.96	11.88	11.85	11.87	11.93	12.01	12.10

Table A.3. Implied Black volatilities for frozen drift prices.

	2.5 %	3.0 %	3.5 %	4.0 %	4.5 %	5.0 %	5.5 %	6.0 %	6.5 %	7.0 %
T_1	0.288	0.274	0.240	0.175	0.116	0.080	0.057	0.042	0.032	0.025
T_2	0.448	0.436	0.412	0.366	0.299	0.230	0.175	0.135	0.107	0.087
T_3	0.502	0.488	0.458	0.406	0.338	0.268	0.209	0.163	0.129	0.103
T_4	0.587	0.575	0.553	0.514	0.456	0.386	0.318	0.257	0.206	0.166
T_5	0.594	0.583	0.560	0.520	0.461	0.393	0.326	0.265	0.213	0.172
T_6	0.619	0.610	0.592	0.558	0.507	0.444	0.376	0.312	0.256	0.208
T_7	0.600	0.592	0.575	0.542	0.493	0.432	0.366	0.304	0.248	0.202
T_8	0.592	0.587	0.575	0.549	0.506	0.450	0.387	0.324	0.267	0.218
T_9	0.554	0.551	0.540	0.517	0.477	0.423	0.361	0.301	0.245	0.198

Table A.4. Standard error $\times 10^{-1}$ of the Monte Carlo simulation for the full SDE prices.

	2.5 %	3.0 %	3.5 %	4.0 %	4.5 %	5.0 %	5.5 %	6.0 %	6.5 %	7.0 %
T_1	0.006	0.019	0.047	0.080	0.057	0.038	0.027	0.019	0.015	0.012
T_2	0.011	0.031	0.070	0.120	0.211	0.193	0.156	0.121	0.094	0.075
T_3	0.023	0.060	0.120	0.182	0.316	0.299	0.256	0.209	0.168	0.135
T_4	0.004	0.010	0.020	0.033	0.043	0.062	0.060	0.054	0.046	0.039
T_5	0.031	0.078	0.152	0.237	0.304	0.533	0.513	0.463	0.403	0.343
T_6	0.048	0.123	0.245	0.394	0.529	0.611	0.982	0.925	0.832	0.726
T_7	0.047	0.118	0.234	0.375	0.500	0.576	0.912	0.861	0.776	0.679
T_8	0.015	0.039	0.081	0.135	0.188	0.226	0.366	0.355	0.326	0.290

Table A.5. Standard error $\times 10^{-5}$ of the Monte Carlo simulation for the difference in caplet prices between the strong Taylor and the full SDE method.

	2.5 %	3.0 %	3.5 %	4.0 %	4.5 %	5.0 %	5.5 %	6.0 %	6.5 %	7.0 %
T_1	0.002	0.006	0.014	0.024	0.017	0.012	0.008	0.006	0.004	0.004
T_2	0.003	0.007	0.016	0.027	0.047	0.043	0.035	0.027	0.021	0.017
T_3	0.004	0.009	0.019	0.028	0.050	0.047	0.040	0.033	0.027	0.021
T_4	0.004	0.010	0.021	0.034	0.045	0.078	0.075	0.066	0.057	0.048
T_5	0.006	0.015	0.028	0.044	0.057	0.098	0.094	0.085	0.074	0.063
T_6	0.005	0.013	0.025	0.041	0.055	0.063	0.101	0.095	0.086	0.075
T_7	0.004	0.010	0.020	0.032	0.043	0.049	0.078	0.074	0.067	0.058
T_8	0.002	0.004	0.009	0.014	0.020	0.024	0.039	0.038	0.035	0.031

Table A.6. Standard error $\times 10^{-3}$ of the Monte Carlo simulation for the difference in caplet prices between the frozen drift and the full SDE method.

Appendix B. Swaption prices and standard errors.

In this appendix we report the swaption prices for the three methods and the standard error of the Monte Carlo simulation for the swaption prices corresponding to the full SDE for the LIBOR rates; we point out that once again the standard errors are very similar in all three methods. We also report the standard errors corresponding to the simulation of the difference

in swaption prices. In all the simulations we have used 1 million paths and 200 time steps.

	2.5 %	3.0 %	3.5 %	4.0 %	4.5 %	5.0 %	5.5 %	6.0 %	6.5 %
1Y–12M	15.46	3.85	1.12	0.38	0.15	0.07	0.03	0.02	0.01
1Y–18M	31.35	8.29	2.41	0.80	0.30	0.13	0.06	0.03	0.02
1Y–24M	48.32	13.40	3.92	1.28	0.47	0.19	0.09	0.04	0.03
1Y–30M	70.27	20.44	6.04	1.95	0.70	0.28	0.12	0.06	0.03
2Y–12M	38.63	13.84	4.89	1.82	0.72	0.31	0.14	0.07	0.03
2Y–18M	63.07	23.11	8.19	3.01	1.18	0.49	0.22	0.10	0.05
2Y–24M	86.91	32.11	11.34	4.13	1.59	0.65	0.28	0.13	0.06
2Y–30M	114.35	42.73	15.05	5.41	2.05	0.83	0.35	0.16	0.07

Table B.1. Swaption prices for the full SDE method.

	2.5 %	3.0 %	3.5 %	4.0 %	4.5 %	5.0 %	5.5 %	6.0 %	6.5 %
1Y–12M	15.46	3.85	1.12	0.38	0.15	0.07	0.03	0.02	0.01
1Y–18M	31.34	8.29	2.41	0.80	0.30	0.13	0.06	0.03	0.02
1Y–24M	48.32	13.40	3.92	1.28	0.47	0.19	0.09	0.04	0.03
1Y–30M	70.27	20.44	6.04	1.95	0.70	0.28	0.12	0.06	0.03
2Y–12M	38.64	13.84	4.90	1.82	0.72	0.31	0.14	0.07	0.03
2Y–18M	63.08	23.12	8.19	3.01	1.18	0.49	0.22	0.10	0.05
2Y–24M	86.93	32.12	11.35	4.13	1.59	0.65	0.28	0.13	0.06
2Y–30M	114.37	42.74	15.06	5.41	2.05	0.83	0.35	0.16	0.07

Table B.2. Swaption prices for the strong Taylor method

	2.5 %	3.0 %	3.5 %	4.0 %	4.5 %	5.0 %	5.5 %	6.0 %	6.5 %
1Y–12M	15.41	3.84	1.12	0.38	0.15	0.06	0.03	0.02	0.01
1Y–18M	31.24	8.26	2.40	0.80	0.30	0.13	0.06	0.03	0.02
1Y–24M	48.14	13.34	3.90	1.27	0.47	0.19	0.09	0.04	0.03
1Y–30M	70.00	20.34	6.01	1.94	0.70	0.28	0.12	0.06	0.03
2Y–12M	38.49	13.78	4.87	1.81	0.72	0.30	0.14	0.07	0.03
2Y–18M	62.83	23.00	8.15	3.00	1.17	0.49	0.22	0.10	0.05
2Y–24M	86.60	31.97	11.29	4.10	1.58	0.65	0.28	0.13	0.06
2Y–30M	113.99	42.55	14.98	5.38	2.04	0.82	0.35	0.16	0.07

Table B.3. Swaption prices for the frozen drift method.

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	2.5 %	3.0 %	3.5 %	4.0 %	4.5 %	5.0 %	5.5 %	6.0 %	6.5 %
1Y–12M	0.047	0.027	0.017	0.012	0.009	0.008	0.007	0.006	0.006
1Y–18M	0.080	0.047	0.029	0.019	0.014	0.011	0.010	0.009	0.008
1Y–24M	0.114	0.068	0.041	0.026	0.018	0.014	0.012	0.011	0.010
1Y–30M	0.153	0.091	0.054	0.034	0.023	0.017	0.014	0.012	0.011
2Y–12M	0.076	0.050	0.032	0.021	0.014	0.010	0.007	0.006	0.005
2Y–18M	0.117	0.077	0.049	0.032	0.021	0.015	0.011	0.008	0.006
2Y–24M	0.156	0.104	0.066	0.042	0.028	0.019	0.013	0.010	0.008
2Y–30M	0.196	0.131	0.083	0.052	0.034	0.023	0.016	0.011	0.009

Table B.4. Standard error of the Monte Carlo simulation for the full SDE prices.

	2.5 %	3.0 %	3.5 %	4.0 %	4.5 %	5.0 %	5.5 %	6.0 %	6.5 %
1Y–12M	0.045	0.029	0.018	0.011	0.007	0.005	0.004	0.003	0.002
1Y–18M	0.053	0.036	0.022	0.014	0.009	0.006	0.004	0.003	0.002
1Y–24M	0.013	0.008	0.005	0.004	0.003	0.002	0.002	0.001	0.001
1Y–30M	0.098	0.074	0.048	0.030	0.019	0.013	0.009	0.006	0.004
2Y–12M	0.049	0.041	0.029	0.020	0.014	0.010	0.007	0.005	0.004
2Y–18M	0.146	0.128	0.093	0.064	0.044	0.030	0.022	0.016	0.011
2Y–24M	0.236	0.210	0.153	0.105	0.071	0.049	0.035	0.025	0.018
2Y–30M	0.271	0.246	0.181	0.124	0.084	0.058	0.040	0.029	0.021

Table B.5. Standard error $\times 10^{-4}$ of the Monte Carlo simulation for the difference in swaption prices between the strong Taylor and the full SDE method.

	2.5 %	3.0 %	3.5 %	4.0 %	4.5 %	5.0 %	5.5 %	6.0 %	6.5 %
1Y–12M	0.091	0.059	0.036	0.023	0.016	0.012	0.009	0.008	0.007
1Y–18M	0.170	0.118	0.073	0.047	0.031	0.022	0.017	0.013	0.011
1Y–24M	0.267	0.193	0.122	0.077	0.051	0.035	0.025	0.019	0.015
1Y–30M	0.368	0.281	0.180	0.115	0.075	0.051	0.036	0.026	0.020
2Y–12M	0.176	0.150	0.107	0.074	0.051	0.036	0.026	0.019	0.014
2Y–18M	0.275	0.242	0.176	0.121	0.083	0.058	0.041	0.030	0.022
2Y–24M	0.351	0.313	0.229	0.157	0.107	0.074	0.052	0.038	0.028
2Y–30M	0.388	0.353	0.260	0.179	0.121	0.083	0.059	0.042	0.030

Table B.6. Standard error $\times 10^{-3}$ of the Monte Carlo simulation for the difference in swaption prices between the frozen drift and the full SDE method.

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